Global monopoles, cosmological constant and maximal mass conjecture

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Abstract

We consider global monopoles as well as black holes with global monopole hair in Einstein-Goldstone model with a cosmological constant in four spacetime dimensions. Similar to the $\Lambda = 0$ case, the mass of these solutions defined in the standard way diverges. We use a boundary counterterm subtraction method to compute the mass and action of $\Lambda \neq 0$ configurations. The mass of the asymptotically de Sitter solutions computed in this way turns out to take positive values in a specific parameter range and, for a relaxed set of asymptotic boundary conditions, yields a counterexample to the maximal mass conjecture.

1 Introduction

Although there is now strong evidence that the universe has a positive cosmological constant [1], there is also much interest in studying theories of gravity with a negative cosmological constant. This interest followed the anti-de Sitter/conformal field theory conjecture, which proposes a correspondence between physical effects associated with gravitating fields propagating in anti-de Sitter (AdS) spacetime and those of a conformal field theory (CFT) on the boundary of AdS spacetime [2, 3]. The results in the literature suggest also the existence of a de Sitter (dS) version of this conjecture which has a number of similarities with the AdS/CFT correspondence, although many details and interpretations remain to be clarified (see [4] for a recent review and a large set of references on these problems).

In view of these developments, an examination of the classical solutions of gravitating fields in spacetimes with a cosmological constant seems appropriate.

The case of a Goldstone scalar field represents a particularly interesting model. Gravitating global monopoles without a cosmological constant were first discussed in [5, 6]. These topological defects are predicted in unified theories and may appear in cosmological phase transitions in the early universe. The gravitating global monopoles have a negative mass and a deficit angle depending on the vacuum expectation value (vev) of the scalar Goldstone field and the gravitational coupling. For sufficiently high enough values of the vev, the solutions have a cosmological horizon [7, 9]. These solutions were named (after their string counterparts [10]) "supermassive monopoles".

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In [11] and [12] the basic features of global monopoles in both asymptotically dS and AdS spacetimes have been studied. However, the issue of mass computation of these solutions has not been addressed in the literature, nor the question of cosmological configurations with a black hole event horizon.

In this paper, we reinvestigate global monopoles in (A)dS space-time, special attention being paid to the solutions' mass and action computation. For $\Lambda > 0$, we construct global monopoles in de Sitter space in- and outside the cosmological horizon, which has to our knowledge not been done so far. In addition, we also study the corresponding cosmological solutions with a black hole horizon, which have up to now only been studied for $\Lambda = 0$ [7] (see also [8] for a discussion of the status of no hair conjecture for solutions with a cosmological horizon).

Although the solutions we discuss are still asymptotically (A)dS, their standard gravitational mass diverges, similar to the $\Lambda = 0$ case. However, asymptotically AdS solutions with a diverging ADM mass have been considered recently by some authors, mainly for a scalar field in the bulk (see e.g. [13]-[18]). In this case it was possible to relax the standard asymptotic conditions without loosing the original symmetries, but modifying the charges in order to take into account the presence of matter fields and to find a well-defined mass of the solutions.

In this paper we propose a similar mass computation for the case of a scalar field with a spontaneously broken internal $O(3)$ symmetry. This is relevant especially for the case of a positive cosmological constant. The authors of [19] conjectured that "any asymptotic dS spacetime with mass greater than dS has a cosmological singularity", which is known in the literature as the *maximal mass conjecture*. Roughly speaking, it means that the conserved mass of any physically reasonable asymptotically dS spacetime must be negative (*i.e.* less than the zero value of the pure dS_4 spacetime). Here we argue that this is valid for a rather limited set of dS boundary conditions, the dS global monopoles providing an explicit counterexample since their mass may take positive values.

Our paper is organized as follows: in Section 2, we give the model including the equations of motion and the boundary conditions. In Section 3 we present our numerical results, while in Section 4, we address the problem of mass and action definition for global monopoles in (A)dS. Section 5 contains our conclusions.

2 The Model

We consider the following action principle

$$I = \int_M d^4x \sqrt{-g}\left(\frac{1}{16\pi G}(R - 2\Lambda) + L_m\right) - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{|h|}K, \tag{1}$$

with the matter Lagrangian

$$L_m = -\frac{1}{2}\partial_\mu\Phi^a\partial^\mu\Phi^a - \frac{\lambda}{4}(\Phi^a\Phi^a - \eta^2)^2, \tag{2}$$

which describes a Goldstone triplet Φ^a, $a = 1, 2, 3$, minimally coupled to Einstein gravity with a cosmological constant Λ. G is Newton's constant, λ the self-coupling constant of the Goldstone field and η the vev of the Goldstone field.

The last term in (1) is the Hawking-Gibbons surface term [20], which is required in order to have a well-defined variational principle, K being the trace of the extrinsic curvature for the boundary ∂M and h the induced metric of the boundary.

For the metric, the spherically symmetric Ansatz in Schwarzschild-like coordinates reads :

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - A^2(r)N(r)dt^2 , \tag{3}$$

with

$$N = 1 - \frac{2m(r)}{r} - \frac{\Lambda}{3}r^2, \tag{4}$$

while for the Goldstone field with a unit winding number, we choose the usual hedgehog Ansatz [5]

$$\Phi^a = \phi(r)e_r{}^a , \tag{5}$$

where e_r is a unit radial vector.

2.1 The equations

Varying (1) with respect to the metric, we obtain the Einstein equations which can be combined to give two first order differential equations for A and m:

$$m' = 4\pi G \left(Nr^2\phi'^2 + 2\phi^2 + r^2 V(\phi) \right) , \tag{6}$$

$$A' = 8\pi G r A \phi'^2. \tag{7}$$

Variation with respect to the matter field leads to the Euler-Lagrange equation for the Goldstone scalar:

$$(ANr^2\phi')' = A \left(2\phi + \frac{1}{2}\frac{\partial V(\phi)}{\partial \phi} \right), \tag{8}$$

with the potential $V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2$ and the prime denotes the derivative with respect to r. Note that the equations have the same structure as for the $\Lambda = 0$ case [5, 6]. The cosmological constant just appears in the relation defining the metric function $N(r)$.

2.2 Boundary conditions

In order to solve the system of equations (6)-(8) we have to impose appropriate boundary conditions for the metric and scalar field. We start by imposing the Goldstone scalar to approach asymptotically the vev, $\Phi^a\Phi^a(r \to \infty) \to \eta^2$, which assures a vanishing scalar potential.

For the metric we consider a weakened set of asymptotic boundary conditions as compared to the standard choice in the literature. In the AdS case, following [21], we suppose that one can attach a boundary, $\mathcal{I} \cong R \times S^2$ to M such that $\tilde{M} = M \cup \mathcal{I}$ is a manifold with boundary. On $\tilde{M}$, there is a smooth metric $\tilde{g}_{ab}$ and a smooth function Ω such that $g_{ab} = \Omega^{-2}\tilde{g}_{ab}$, and such that $\Omega = 0$ and $\tilde{n}_a \equiv \tilde{\nabla}_a\Omega \neq 0$ at points of $\mathcal{I}$ (a possible choice for the line element (3) is $\Omega = 1/r$). The metric $\tilde{h}_{ab}$ on $\mathcal{I}$ induced by $\tilde{g}_{ab}$ is in the conformal class of the Einstein static universe, $\tilde{h}_{ab} dx^a dx^b = e^\omega[-dt^2 + \ell^2 d\sigma^2]$, where $d\sigma^2$ is the line element of the unit sphere S^2, and ω is some smooth function (with $\ell^2 = 3/|\Lambda|$) [1].

A similar set of boundary conditions is considered for solutions with a positive cosmological constant. In this case, the metric $\tilde{h}_{ab}$ on the boundary induced by $\tilde{g}_{ab}$ is in the conformal class of the Euclidean Einstein static universe, $\tilde{h}_{ab} dx^a dx^b = e^\omega[dt^2 + \ell^2 d\sigma^2]$, i.e. $\mathcal{I}$ is a spacelike boundary.

The field equations (6)-(8) imply the following behavior as $r \to \infty$

$$
\begin{aligned}
m(r) &= 8\pi G\eta^2 r + M_0 + \frac{144\pi G\eta^4\lambda}{(3\eta^2\lambda - 2\Lambda)^2}\frac{1}{r} + O(1/r^3), \\
A(r) &= 1 - \frac{288\pi G\eta^2}{(3\lambda\eta^2 - 2\Lambda)^2}\frac{1}{r^4} + O(1/r^5), \\
\phi(r) &= \eta + \frac{6\eta}{2\Lambda - 3\eta^2\lambda}\frac{1}{r^2} + \frac{18\eta^3(\lambda(96\pi G\eta^2 - 9) - 64\pi G\Lambda)}{(3\lambda\eta^2 - 2\Lambda)^2(4\Lambda + 3\eta^2\lambda)}\frac{1}{r^4} + O(1/r^5).
\end{aligned}
\tag{9}
$$

Therefore the function $m(r)$ whose asymptotically value usually gives the mass of solution, presents a linear divergence as $r \to \infty$ (a similar linear divergence in the ADM mass was found also in various gauged supergravities with a dilaton field possessing a nontrivial potential approaching a constant negative value at infinity,

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[1] These general metric boundary conditions hold also for less symmetric solutions, e.g. axially symmetric global monopoles which may exist for a winding number $n \neq 1$.

see *e.g.* [13], [16]). However, this is consistent with the asymptotic set of boundary conditions we considered [2]. For particle-like solutions, $r = 0$ is a regular origin, the following set of initial conditions being satisfied:

$$
\begin{aligned}
m(r) &= \frac{1}{3}\pi G(12\phi_1^2 + \lambda\eta^4)r^3 + O(r^5), \\
A(r) &= A_0 + 4\pi G\phi_1^2 A_0 r^2 + O(r^4), \\
\phi(r) &= \phi_1 r + \frac{\phi_1}{16}\left(-3\eta^2\lambda + 16\pi G(9\phi_1^2 + \eta^4\lambda) + 8\lambda\right)r^3 + O(r^4),
\end{aligned}
\tag{10}
$$

where ϕ_1 and A_0 are free parameters to be determined numerically.

We will also consider solutions with an event horizon located for some $r = r_h > 0$ ("black holes inside global monopoles"). The behaviour of the functions close to $r = r_h$ reads:

$$
\begin{aligned}
m(r) &= m(r_h) + m'(r_h)(r - r_h) + O(r - r_h)^2, \\
A(r) &= A(r_h) + A'(r_h)(r - r_h) + O(r - r_h)^2, \\
\phi(r) &= \phi(r_h) + \phi'(r_h)(r - r_h) + O(r - r_h)^2,
\end{aligned}
\tag{11}
$$

with

$$
m_h = \frac{r_h}{2}\left(1 - \frac{\Lambda}{3}r_h^2\right), \quad m'(r_h) = 4\pi G\left(2\phi(r_h)^2 + r_h^2 V(\phi(r_h))\right),
\tag{12}
$$

$$
A'(r_h) = 8\pi G r_h A(r_h)\phi'(r_h)^2, \quad \phi'(r_h) = \frac{1}{N'(r_h)r_h^2}\left(2\phi(r_h) + \frac{1}{2}\left.\frac{\partial V(\phi)}{\partial\phi}\right|_{r_h}\right),
\tag{13}
$$

where $\phi(r_h)$ and $A(r_h)$ are free parameters and $N'(r_h) = 1/r_h - \Lambda r_h/3$.

The dS solutions we consider in this paper possess a cosmological horizon located at $r = r_c > 0$, where an expansion similar to (11) is valid (replacing r_h by r_c).

By going to the Euclidean section (or by computing the surface gravity) one finds the Hawking temperature of the solutions with an horizon at $r = \tilde{r}$ (where $\tilde{r} = r_h$ or $\tilde{r} = r_c$) to be:

$$
T = \frac{1}{\beta} = \frac{A(\tilde{r})|N'(\tilde{r})|}{4\pi} .
\tag{14}
$$

3 Numerical results

To perform numerical computations and order-of-magnitude estimations, it is useful to have a new set of dimensionless variables. Therefore we introduce the following dimensionless coordinate x, field H and coupling constant α:

$$
x = \eta r, \quad H(x) = \frac{\phi(r)}{\eta}, \quad \alpha^2 = 4\pi G\eta^2,
\tag{15}
$$

and replace $\Lambda/\eta^2 \to \Lambda$. We also introduce a new function μ in order to subtract the linear part in (9):

$$
\mu(x) = \eta\left(m(r) - 8\pi G\eta^2 r\right) .
\tag{16}
$$

As we will argue in Section 4, the mass of the solution is determined by the parameter M_0 which enters the asymptotics of $m(r)$; its dimensionless value is M_0/α^2.

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[2]Note that, similar to other $\Lambda \neq 0$ solutions with a linearly divergent mass function (see *e.g.* [22]), it is possible to redefine the radial coordinate $r \to (1 - 8\pi G\eta^2)^{1/2}\bar{r}$, such that the metric at infinity is (A)dS with an angular deficit. However, this transformation should be supplemented by a further rescaling $t \to (1 - 8\pi G\eta^2)^{-1/2}\bar{t}$, and would imply a rescaling in the boundary metric as well.

3.1 Global monopoles in (A)dS

We start by discussing the globally regular solutions of the system of equations (6)-(8). First, we will recall the features of the $\Lambda = 0$ solutions, studied in [5, 6, 7]. For $0 \leq \alpha \leq \sqrt{1/2}$ regular solutions without horizons exist. After a suitable rescaling of the coordinates, one finds that these have a solid deficit angle $\varphi_{deficit} = 2\alpha^2$ (compare (9)). For $\sqrt{1/2} \leq \alpha \leq \sqrt{3/2}$ the solutions develop a horizon at some α-dependent value $x = x_{sm}(\alpha)$. These solutions are the so-called "supermassive" monopoles studied in [7]. For $1 \leq \alpha \leq \sqrt{3/2}$ the metric function $\mu(x)$ and thus also $N(x)$ develop oscillations in the region outside the horizon. For $\alpha > \sqrt{3/2}$ no static solutions exist at all.

Considering asymptotically (A)dS solutions, we observe that the $\Lambda = 0$ configurations get progressively deformed by a nonzero cosmological constant. Here we consider the cases $\alpha = 0.5$ and $\alpha = 1.0$ only, but we believe that these represent the generic features of the solutions.

The results for these two values of α are given in Figs. 1 and 2. For $\alpha = 0.5$ and $\Lambda < 0$ the global monopole exists for arbitrary values of the negative cosmological constant. The parameter M_0 characterizing the mass of the solution is negative. In Fig.1 the data is given for $|\Lambda| \leq 0.2$, but in [12] it is demonstrated that M_0 stays negative for any $\Lambda < 0$ and approaches zero in the limit $\Lambda \to -\infty$. This feature seems to occur irrespectively of the value of α.

For $\alpha = 0.5$ and $\Lambda > 0$ the solution develops a cosmological horizon at some Λ-dependent value of the radial coordinate $x = x_c$. With the boundary conditions discussed above, the procedure is to integrate separately between the origin and cosmological horizon and from the cosmological horizon to infinity, matching the solutions at the cosmological horizon. In Fig.1, we plot M_0, the value of the cosmological horizon x_c, the Hawking temperature at the horizon $T(x_c)$ and the value of the scalar field function H at the horizon $H(x_c)$ as functions of Λ. Again, the mass parameter M_0 turns out to be always negative and decreases with increasing Λ.

The situation for $\alpha = 1.0$ is more subtle. In this case the $\Lambda = 0$ solution possesses a "supermassive monopole" horizon at $x = x_{sm}$. Decreasing Λ slightly from zero, we observe that a second zero of the function $N(r)$ develops at some $\tilde{x}_{sm} \gg x_{sm}$. Continuing to decrease Λ our numerical analysis reveals that the values $\tilde{x}_{sm}$, x_{sm} approach each other and coincide at a critical value of Λ (for $\alpha = 1.0$ we find for $\Lambda \approx -0.045$, $x_{sm} = \tilde{x}_{sm} \approx 4.5$). Then for even lower value of Λ the solution has no horizons. Again the mass parameter M_0 is negative. For $\Lambda > 0$ the solution has a single, cosmological horizon at $x = x_c$. This zero of the metric function $N(x)$ is in fact the continuation of the one associated with the supermassive $\Lambda = 0$ solution. The parameter M_0, the different horizons' values and the temperature at these horizons are given in Fig.2.

The profiles of the metric and matter functions for a typical global monopole solution in de Sitter space are given in Fig.3. The profiles of the functions $N(x)$ and $\mu(x)$ for three characteristic values of Λ are given in Fig.4.

We have further noticed that the oscillations of the function $N(x)$ (which are very small for $\Lambda = 0$ and $\alpha = 1.0$) are amplified for increasing Λ. Therefore it becomes difficult to accurately determine the parameter M_0 for $\Lambda > 0.1$. However, our numerical results give strong evidence that it is always negative.

3.2 Black holes inside global monopoles

We study also black hole solutions inside global monopoles. For this, we have to replace the boundary conditions for the global monopoles by boundary conditions at a regular horizon located at $x = x_h$. We find that the corresponding black hole solutions share many features with their regular counterparts discussed above. Since there are now three continuous parameter (namely x_h, Λ, α) to vary, we restrict our numerical investigations to the case $x_h = 0.5$ and to $\alpha = 0.5$ and $\alpha = 1.0$.

In Fig. 5, we give our results for $\alpha = 0.5$ and $-0.3 \leq \Lambda \leq 0.3$. There is shown also the mass parameter M_0 which in this case is again negative and decreases monotonically with increasing cosmological constant (this was checked for larger values of Λ than the set shown in the figure). In asymptotically dS spacetime two horizons occur (an event horizon at $x = x_h$ and a cosmological horizon at $x = x_c$). The Hawking temperatures at the two horizons $T(x_c)$ and $T(x_h)$ are also given in Fig.5.

For increasing Λ the two horizons approach each other (the cosmological horizon moves to smaller x) and for $\Lambda \sim 0.4$ we notice that the function $\mu(r)$ becomes oscillating. As far as our numerical simulation confirms,

the mean values of the oscillations is negative but it becomes impossible to compute the mass of the solution by the method underlined above.

We find also that for small values of α (e.g. $\alpha = 0.5$) the evolution of the function $N(x)$ with varying Λ shows no occurrence of a pronounced local minimum. This is shown in Fig. 6. This figure however reveals that this changes for large enough values of α. For $\alpha = 1$, indeed, the function N develops a local minimum for $\Lambda \geq -0.2$. While Λ increases the minimal value of N becomes smaller and smaller, approaching zero for some x_m and then taking even negative values. Thus, for $-0.05 \leq \Lambda \leq 0$ the function N has three zeros (see Fig.6). We have found that the largest of these zeros tends to infinity in the $\Lambda \rightarrow 0$ such that the solution has two zeros for $\Lambda \geq 0$.

Different from globally regular solutions, the mass parameter M_0 of the cosmological black hole configurations may take positive values as well, for both signs of Λ. The occurrence of black hole solutions with $M_0 > 0$ of the gravitating Goldstone model has been already noticed in [23] for $\Lambda = 0$. In Fig. 7 we show the profiles of two typical dS solution with $\Lambda = 0.025$, $\alpha = 0.5$ and two different values of x_h. Clearly, the solution with $x_h = 1.0$ has $M_0 > 0$ (μ approaches a positive value for $x \rightarrow \infty$). We find that configurations with mass parameter $M_0 \geq 0$ exist also in the $\Lambda < 0$ case, for large enough values of the event horizon radius.

4   Global monopole mass and action

4.1   AdS solutions

An important problem of AdS space concerns the definition of action and conserved charges of asymptotically AdS solutions. As concerning the mass, the generalization of Komar's formula in this case is not straightforward and requires the further subtraction of a reference background configuration in order to render a finite result. This problem was addressed for the first time in the 1980s, with different approaches (see for instance [21] for a recent review).

In the Brown-York approach [24], one defines a quasilocal stress-tensor through the variation of the gravitational action

$$T_{ab} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h^{ab}}. \tag{17}$$

The conserved charge associated with the time translation symmetry of the boundary metric is the mass of spacetime. However, T_{ab} diverges as the boundary is pushed to infinity, and hence a background subtraction is again necessary. In [25] this procedure has been improved by regulating the boundary stress tensor through the introduction of an appropriate boundary counterterm. This method does not require the introduction of a somewhat artificial reference background and it has become the standard approach when applied to AdS/CFT, as the boundary counterterms have a natural interpretation in terms of conventional field theory counterterms that show up in the dual CFT.

As found in [25], the following counterterms are sufficient to cancel divergences in four dimensions, for vacuum solutions with a negative cosmological constant $\Lambda = -3/\ell^2$ (in this Section we do not take the rescaling (15))

$$I_{\text{ct}}^0 = -\frac{1}{8\pi G} \int_{\partial \mathcal{M}} d^3 x \sqrt{-h} \left[\frac{2}{\ell} + \frac{\ell}{2} \mathcal{R} \right], \tag{18}$$

where $\mathcal{R}$ is the Ricci scalar for the boundary metric h.

Using this counterterm, one can construct a divergence-free boundary stress tensor $T_{\mu\nu}$ from the total action $I = I_{\text{bulk}} + I_{\text{surf}} + I_{\text{ct}}^0$ by defining

$$T_{ab}^{(0)} = \frac{1}{8\pi G} (K_{ab} - K h_{ab} - \frac{2}{\ell} h_{ab} + \ell E_{ab}), \tag{19}$$

where E_{ab} is the Einstein tensor of the boundary metric, $K_{ab} = -1/2(\nabla_a n_b + \nabla_b n_a)$ is the extrinsic curvature, n^a being an outward pointing normal vector to the boundary.

If ξ^a is a Killing vector generating an isometry of the boundary geometry, there should be an associated conserved charge. We suppose that the boundary geometry is foliated by spacelike surfaces Σ with metric σ_{ij}

$$h_{ab}dx^a dx^b = -N_\Sigma^2 dt^2 + \sigma_{ij}(dx^i + N_\sigma^i dt)(dx^j + N_\sigma^j dt). \tag{20}$$

The conserved charge associated with time translation $\partial/\partial t$ symmetry of the boundary metric is the mass of spacetime

$$\mathbf{M} = \int_\Sigma d^2 x \sqrt{\sigma} N_\Sigma \epsilon, \tag{21}$$

where $\epsilon = u^a u^b T_{ab}$ is the proper energy density and u^a is a timelike unit normal to Σ.

The presence of the additional matter fields in the bulk action brings the potential danger of having divergent contributions coming from both the gravitational and matter action [26]. Various examples of asymptotically AdS solutions whose action and mass cannot be regularized by employing only the counterterm (18) have been presented in the literature. This is also the case of the asymptotically AdS solutions of the Goldstone model, whose mass and action are not regularized by the counterterm (18). To evaluate the Euclidean bulk action, we use the Killing identity $\nabla^a \nabla_b K_a = R_{bc} K^c$, for the Killing field $K^a = \delta_t^a$, together with the Einstein equation

$$\frac{1}{16\pi G}(R - 2\Lambda) + L_m = \frac{1}{8\pi G} R_t^t, \tag{22}$$

such that it is possible to write the bulk action contribution as a difference of two surface integrals at infinity and on the event horizon, or at $r = 0$ for configurations with a regular origin. Restricting to this last class of solutions, one find from (9) $I_{bulk} = 4\pi\beta(r^3/(8\pi G\ell^2) + M/(8\pi G) + O(1/r))$, with β the periodicity of the Euclidean time. (The black hole solutions have a finite supplementary event horizon contribution). The Gibbons-Hawking boundary term yields $I_{GH} = 4\pi\beta(-3r^3/(8\pi G\ell^2) - (1/(4\pi G) - 4\eta^2)r + 3M/(8\pi G) + O(1/r))$. One can see that the geometric boundary counterterm (18), $I_{ct}^0 = 4\pi\beta(r^3/(4\pi G\ell^2) - (1/(4\pi G) - 2\eta^2)r - M/(4\pi G) + O(1/r))$, fails to regularize the divergencies associated with the bulk scalar field, the solutions' action diverging as $I \sim 8\pi\beta\eta^2 r$. Also, the component $T_t^{t(0)}$ of the boundary stress tensor decays too slow, $T_t^{t(0)} = -2\ell\eta^2/r^2 - M\ell/(4\pi G r^3) + O(1/r^4)$, which implies a divergent mass, as computed according to (21).

However, in such cases, it is still possible to obtain a finite mass and action by allowing I_{ct} to depend not only on the boundary metric h_{ab}, but also on the matter fields. This means that the quasilocal stress-energy tensor (19) also acquires a contribution coming from the matter fields. There are two main prescriptions for calculating these supplementary boundary counterterms. The first of these involves the asymptotic expansion of metric tensor and bulk matter fields near the boundary of spacetime. The matter counterterms are covariant expressions of fields living at the boundary which remove all divergencies of the on-shell action and give finite conserved charges. A systematic development of this method for bulk gravity coupled to scalar fields was given first in [27] (see also [28], [29]).

A sufficient general matter counterterm action for the four dimensional solutions we consider here is given by $I_{ct}^{(\phi)} = \int_{\partial M} d^3 x \sqrt{-h}(Y(\Phi) + C(\Phi)\mathcal{R})$, where $Y(\Phi)$, $C(\Phi)$ are polynomial functions of the Goldstone scalar. We also suppose that η does not enter this generic expression (η will appear only when considering the asymptotic expression (9) of the scalar field). As implied by (9), the large r asymptotics of the matter counterterm action is $I_{ct}^{(\phi)} \sim (Y(\eta)r^3/\ell + 2C(\eta)r/\ell)$. One can see that $Y(\Phi) = 0$ necessarily, while the choice $C(\Phi) = \ell\Phi^a\Phi^a$ removes the linear divergence in the total action.

The second method is based on the Hamilton-Jacobi formalism and was first applied in the AdS/CFT context in [30]. In this approach, one takes into account the holographic principle of flows in the radial direction and defines conjugate momenta to the bulk field variables and the Hamiltonian with respect to the AdS radial coordinate r. Diffeomorphism invariance of the theory constrains the Hamiltonian to vanish. To obtain the Hamilton-Jacobi equation one must rewrite the Hamiltonian constraint in terms of functional derivatives of the on-shell action. The conjugate momenta takes a supplementary contribution due to the counterterm action. The counterterms are determined by solving the Hamilton-Jacobi equation order by order in the metric expansion (see [31] for a review of this method).

The ref. [32] considered asymptotically AdS solutions for a general bosonic action consisting in gravity coupled to a set of scalar and vector fields, in d-spacetime dimensions. The general form of the corresponding boundary counterterms was derived by using the Hamilton-Jacobi formalism. The action principle in [32] is general enough to contain (1) as a particular case. Thus one can use the results derived there for the equations satisfied by $Y(\Phi)$ and $C(\Phi)$. As expected, one finds that the choice $Y(\Phi) = 0$ and $C(\Phi) = \ell\Phi^a\Phi^a$ solves the first order expansion in powers of Φ^a of the general equations in Section 3 of [32][3].

Thus we find that by adding a counterterm of the form

$$I_{ct}^{(\phi)} = \int_{\partial M} d^3x\sqrt{-h}\,\ell\phi^2\mathcal{R} \tag{23}$$

to the expression (18), the linear divergence associated with a divergent asymptotic value of $m(r)$ disappears. This yields a supplementary contribution to (19), $T_{ab}^{(\phi)} = -2\ell\phi^2 E_{ab}$. The nonvanishing components of the resulting boundary stress-tensor $T_{ab} = T_{ab}^{(0)} + T_{ab}^{(\phi)}$ are

$$T_\theta^\theta = T_\varphi^\varphi = \left(\frac{M_0}{8\pi G}\right)\frac{\ell}{r^3} + O\left(\frac{1}{r^4}\right), \quad T_t^t = \left(-\frac{M_0}{4\pi G}\right)\frac{\ell}{r^3} + O\left(\frac{1}{r^4}\right).$$

We remark that, to leading order, this stress tensor is traceless as expected from the AdS/CFT correspondence, since even dimensional bulk theories are dual to odd dimensional CFTs which have a vanishing trace anomaly. Employing the AdS/CFT correspondence, this result can be interpreted (after a suitable rescaling) as the expectation value of the stress tensor in the boundary CFT [33]. The mass of solutions, as computed from (21) is $\mathbf{M} = M_0$.

As stated above, the sum of the counterterms (18) and (23) regularizes the infrared divergencies, such that the contribution from the asymptotic region to the total action is found to be $\beta M_0/G$ (where β is given by (14) for black hole solutions and takes arbitrary values for regular configurations). For black hole solutions, there is also an event horizon contribution from the bulk term, $I_h = \frac{\beta}{4G}(r^2 g^{tt}g^{rr}g_{tt,r})|_{r_h}$. Thus, the standard relation

$$S = \beta\mathbf{M} - I \tag{24}$$

gives an entropy of the black holes with global monopole hair which is one quarter of the event horizon area.

4.2   The mass of dS solutions

The computation of mass in an asymptotically dS spacetime is a more difficult task. This is due to the absence of the spatial infinity and the globally timelike Killing vector in this case.

In [19], a novel prescription was proposed, this obstacle being avoided by computing the quasilocal tensor of Brown and York (augmented by the AdS/CFT inspired counterterms), on the Euclidean surfaces at $\mathcal{I}^\pm$. The conserved charge associated with the Killing vector $\partial/\partial t$ - now spacelike outside the cosmological horizon- was interpreted as the conserved mass. This allows also a discussion of the thermodynamics of the asymptotically dS solutions outside the event horizon, the boundary counterterms regularizing the (tree-level) gravitation action [34]. The efficiency of this approach has been demonstrated in a broad range of examples.

In this approach, the initial action (1) is supplemented again by a boundary counterterm action I_{ct}, depending only on geometric invariants of the boundary metric. In four dimensions, the counterterm expression is (here $\Lambda = 3/\ell^2$) [19, 34]

$$I_{ct}^0 = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}^\pm} d^3x\sqrt{h}\left[\frac{2}{\ell} - \frac{\ell}{2}\mathcal{R}\right], \tag{25}$$

the corresponding boundary stress tensor being

$$T_{\mu\nu}^{(0)} = \frac{1}{8\pi G}(K_{\mu\nu} - Kh_{\mu\nu} - \frac{2}{\ell}h_{\mu\nu} - \ell E_{\mu\nu}). \tag{26}$$

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[3]Note that the boundary counterterms equations derived in [32] include also the geometric terms (18). Also, in the concrete examples discussed there, the scalar field was supposed to vanish asymptotically as $\Phi^a \sim 1/r^{d-3}$.

(Note that these expressions can be obtained from the AdS counterparts (18), (19) [34]). However, we have found that the function $m(r)$ always diverges as $r \to \infty$ for solutions with a positive cosmological constant. As a result, the counterterm (25) fails to regularize the action and mass of Goldstone-dS configurations, the structure of the divergences being similar to the AdS case.

The issue of mass and action renormalization for such cases is considerably less explored for dS asymptotics. However, we find that by supplementing the total action with the dS version of the matter counterterm (23)

$$I_{ct}^{(\phi)} = - \int_{\partial \mathcal{M}^\pm} d^3x \sqrt{h} \, \ell \, \Phi^2 \mathcal{R}, \tag{27}$$

yields a finite action for Goldstone-dS configurations (evaluated at timelike infinity outside the cosmological horizon). This gives also a supplementary contribution to the total boundary stress tensor $T_{ab}^{(\phi)} = 2\ell\phi^2 E_{ab}$.

As a result, the mass of asymptotically dS configurations is $\mathbf{M}=-M_0$, which in the absence of matter fields agrees with the standard value for the Schwarzschild-dS solution (the overall sign-flip in the mass arises from the relative signature change in the boundary metric as compared to AdS case). However, we have found numerically that the dS global monopoles have $M_0 < 0$ for a range of parameters, without the occurance of a cosmological singularity. This implies a positive value of $\mathbf{M}$, contradicting obviously the maximal mass conjecture proposed in [19], which therefore should be supplemented by specifying the set of boundary conditions satisfied by the metric functions at infinity.

5   Conclusions

In this paper we have performed a general study of the properties of the gravitating Goldstone-model solutions with a cosmological constant, in four spacetime dimensions. Both globally regular solutions and configurations with a black hole event horizon have been considered.

The previous results on globally regular solutions in dS space have been extended by constructing the solutions both inside and outside the cosmological horizon. For large enough values of α, these solutions have two horizons, one relating to the horizon of the "supermassive monopole" and one to the cosmological horizon.

Studying black holes inside global monopoles, we have noticed that black holes solutions with up to three zeros of the metric function are possible in AdS space for large enough gravitational coupling and Λ close to zero. The mass parameter of the black hole solutions may take both positive and negative values.

The asymptotic behaviour of the metric in the presence of a Goldstone scalar field is different from that in pure gravity. Similar to the $\Lambda = 0$ case, the mass-energy density of these configurations decreases only like $1/r^2$, such that the total mass energy defined in the usual way is diverging linearly at large distances. The mass of the asymptotically dS solutions computed according to a counterterm subtraction procedure turns out to take positive values, which violates the maximal mass conjecture put forward in [19].

One may argue that this violation is not a surprise, since the maximal mass conjecture has been proposed in the context of pure dS gravity. Its validity has been has been tested for Schwarzschild-dS black holes [34] and topological dS solutions [35]. However, it has also been verified for several different solutions with matter fields, including dilatonic deformation of the dS black holes [35] and Reissner-Nordström-dS black holes [36]. A violation of this conjecture has been noticed in [37] (see also [38]). However, this case requires the presence of a NUT charge and thus a different form of the boundary metric. A review of this subject is presented in [39].

Our results indicate that a more general matter content may allow, however, for configurations whose mass is greater than that of dS spacetime. Here we should emphasize again the central role played in our example by the choice of asymptotic set of boundary conditions satisfied by the metric functions. (This choice was imposed by the particular matter model we have considered.) The cases discussed so far in this context assumed the more restrictive set $g_{tt} \sim -1 + \Lambda r^2/3 + O(1/r)$, $g_{rr} \sim -3/(\Lambda r^2) - 9/(\Lambda^2 r^4) + O(1/r^5)$, or the equivalent expression when replacing the two-sphere with a surface with the same amount of symmetry (the nut-charged case requires a separate discussion). It would be interesting to consider the $\Lambda > 0$ analogous of various asymptotically AdS systems with a divergent ADM mass discussed in the literature and to look for further violations of the maximal mass conjecture.

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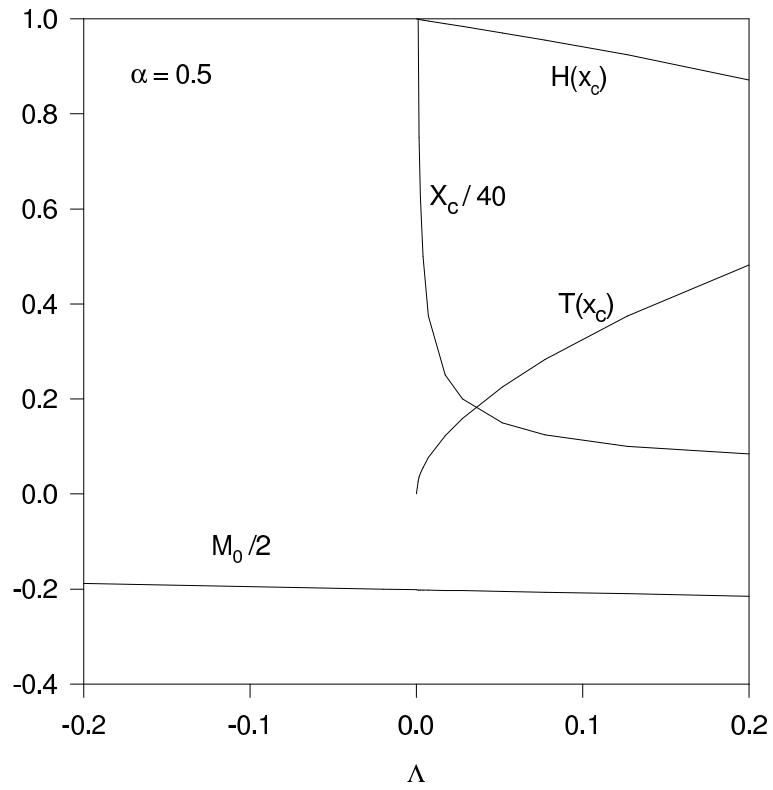
Figure 1: The mass parameter M_0, the value of the cosmological horizon radius x_c, the temperature at the cosmological horizon $T(x_c)$ as well as the value of the scalar field function at the cosmological horizon $H(x_c)$ are shown as functions of the cosmological constant Λ for the global monopoles with $\alpha = 0.5$.
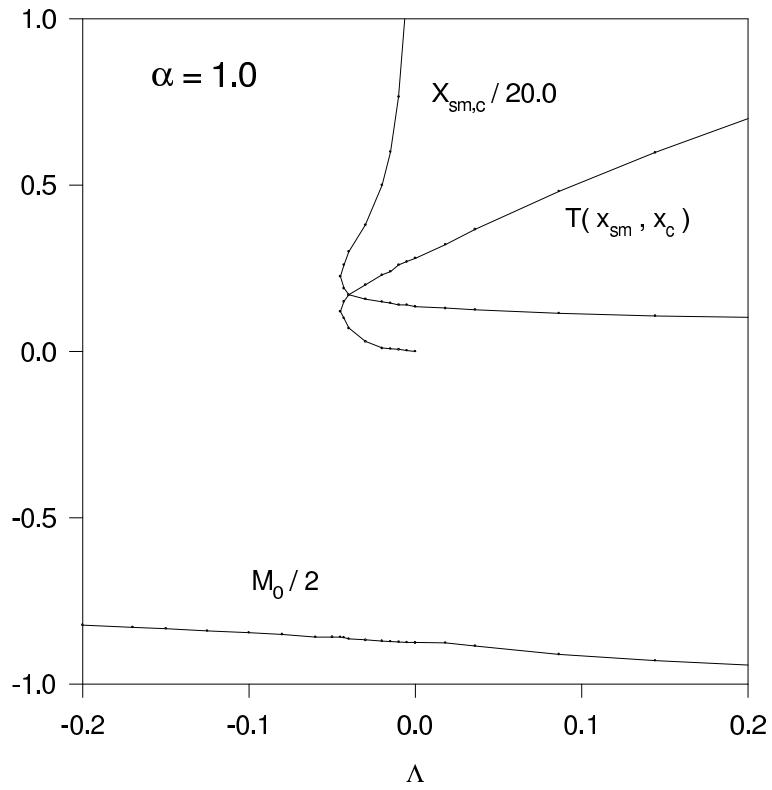
Figure 2: The mass parameter M_0, the value of the cosmological horizon radius x_c, respectively of the supermassive monopole horizon x_{sm} as well as the temperature at the respective horizons $T(x_c, x_{sm})$ are shown as functions of Λ for global monopole solutions with $\alpha = 1.0$.
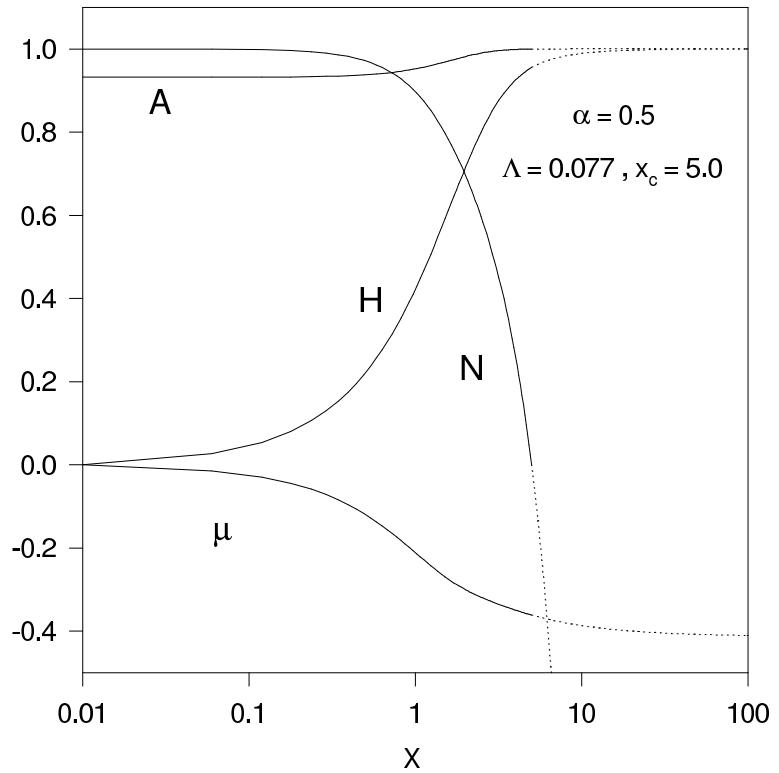
Figure 3: The profiles of the metric function A, N and μ as well as of the scalar field function H are shown for a typical global monopole in de Sitter space with $\alpha = 0.5$ and $\Lambda = 0.077$. The cosmological horizon is located at $x_c = 5.0$.
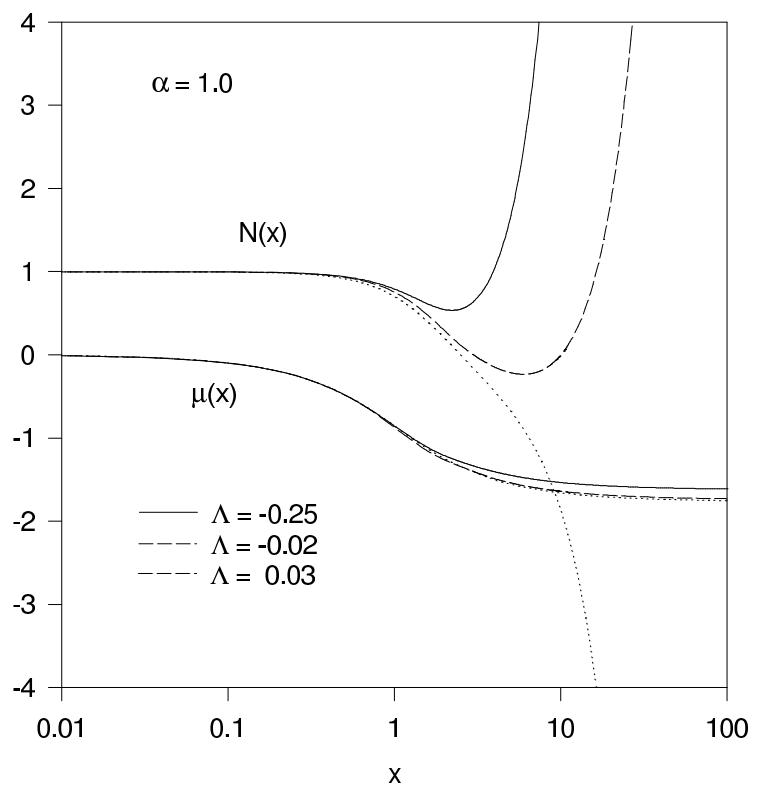
Figure 4: The profiles of the metric function $N(x)$ and $\mu(x)$ are shown for $\alpha = 1.0$ and $\Lambda = -0.25, -0.02$ (AdS solutions) and $\Lambda = 0.03$ (dS solutions).
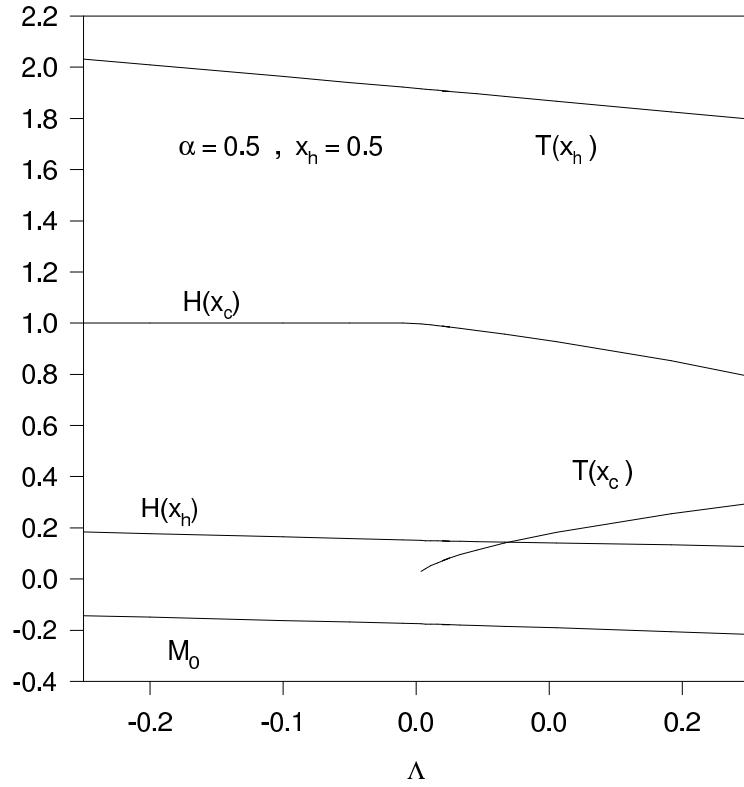
Figure 5: The values of the scalar field function H and the temperature T at the cosmological horizon x_c, $H(x_c)$ and $T(x_c)$ as well as at the black hole horizon x_h, $H(x_h)$ and $T(x_h)$ are shown as functions of Λ for black holes inside global monopoles with $\alpha = 0.5$ and $x_h = 0.5$. The mass parameter M_0 is also shown.
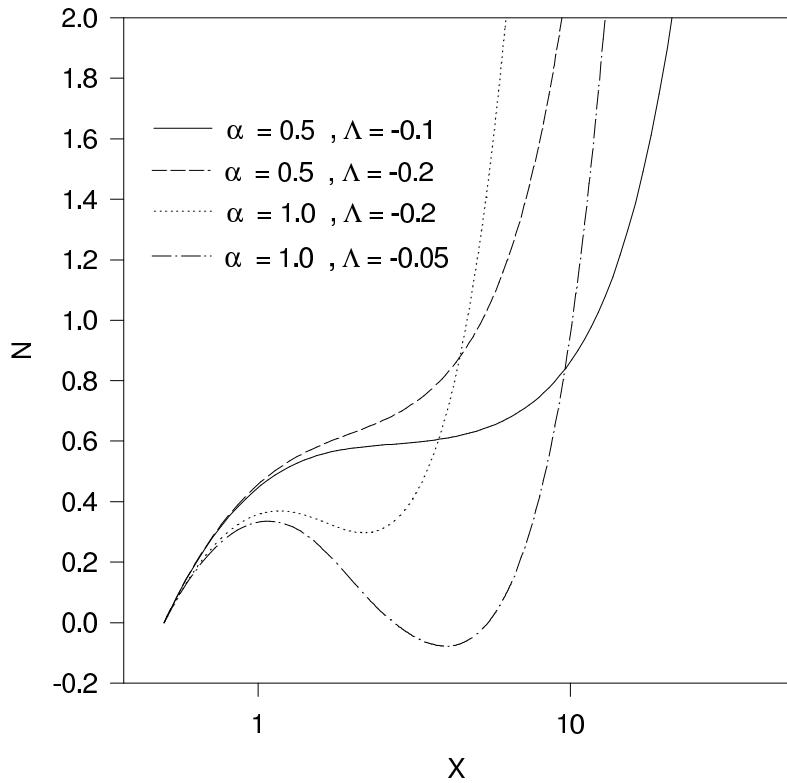
Figure 6: The profiles of the metric function $N(x)$ are shown for different choices of (α, Λ) in AdS space. Note that the solutions with $\alpha = 1.0$ and $\Lambda = -0.05$ possesses 3 horizons.
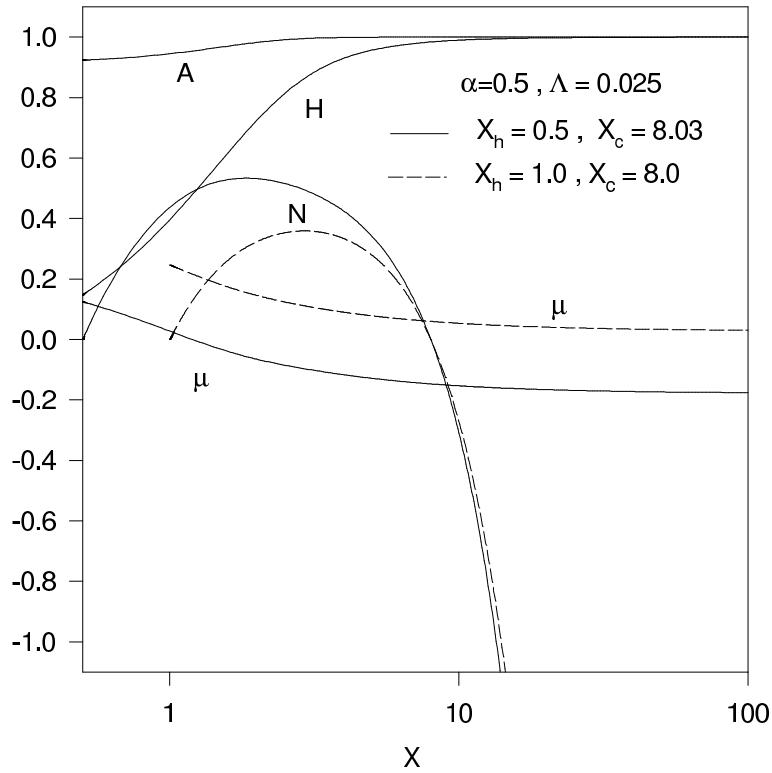
Figure 7: The profiles of the metric functions A, N and μ as well as of the scalar field function H are shown for two typical black holes inside global monopoles in de Sitter space with $\alpha = 0.5$, $\Lambda = 0.025$. Note that the profiles of the functions A and H for $x_h = 1$ are almost identically with those of the $x_h = 0.5$ solution.

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